Income and substitution effect on consumption

Consider a utility function of the following form:

$$u\left(C_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma}$$

For $0 \leq \sigma < 1$, the intertermporal elasticity of substitution (IES), $1/\sigma$, is high and consumption is easily substitutable between time periods. For $\sigma > 1$, the IES is low and the consumer does not like moving consumption between periods. For $\sigma \to 0$, we get that $u\left(C_t\right) \to \log C_t$, which represents the preferences consistent with balanced growth in an RBC model with additively separable utility over consumption and leisure.

Maximizing the NPV of utility subject to a standard budget constraint gives the usual Euler equation (for simplicity, I assume the interest rate is constant):

$$C_t^{-\sigma} = \beta\left(1+r\right) C_{t+1}^{-\sigma}$$

In logs,

$$c_t = c_{t+1} - \frac{1}{\sigma}\log\beta - \frac{1}{\sigma}\log\left(1+r\right) \simeq c_{t+1} - \frac{1}{\sigma}\left(r-\rho\right)$$

where $\beta = 1-\rho$.

First of all, if the interest rate increases, $r \uparrow$, we get unambiguously that consumption growth increases as well, $\left(c_{t+1} - c_t\right) \uparrow$. This is a *substitution effect*: an increase in the interest rate makes consumption tomorrow relatively less expensive compared to consumption today (because saving is more lucrative so to afford the same amount of consumption tomorrow, the consumer needs to sacrifice less consumption today). Thus, the consumer substitutes some consumption today for consumption tomorrow.

Does this mean that consumption today c_t decreases? Not necessarily. In addition to a substitution effect, the consumer also experiences an *income effect* from the increase in interest rates. Assuming she has positive assets, higher interest rates mean higher interest rate income. This effect tends to increase consumption in all periods. The substitution effect tends to increase consumption tomorrow but decrease consumption today. Thus, the net effect on consumption tomorrow is unambiguously positive. The net effect on consumption today may be positive or negative.

Why do we not see the income effect in the consumption Euler equation? Because the Euler equation pins down only the relatively consumption today versus tomorrow, but not the level of consumption. For that, we need the budget constraint. Consider the simplest case: the consumer has some assets at period t but never receives any more income. Solving the dynamic budget constraint forward, we get an intertemporal budget constraint:

$$A_{t+1} = \left(1+r\right)\left(A_t - C_t\right) \Rightarrow A_t = \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s C_{t+s}$$

From the Euler equation, we have $C_{t+1} = \left[\beta\left(1+r\right)\right]^{1/\sigma} C_t \Rightarrow C_{t+s} = \left[\beta\left(1+r\right)\right]^{s/\sigma} C_t$. Substituting this into the budget constraint, we get an explicit expression for consumption at time t.

$$A_t = \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s \left[\beta\left(1+r\right)\right]^{s/\sigma} C_t = \frac{C_t}{1-\frac{\left[\beta(1+r)\right]^{1/\sigma}}{1+r}} \Leftrightarrow C_t = \left(1 - \frac{\left[\beta\left(1+r\right)\right]^{1/\sigma}}{1+r}\right) A_t$$

Or, in logs

$$c_t = \log\left(1 - \frac{[\beta(1+r)]^{1/\sigma}}{1+r}\right) + a_t \simeq a_t - \frac{[\beta(1+r)]^{1/\sigma}}{1+r}$$

Taking a derivative with respect to the interest rate r allows to evaluate the net effect of income and substitution effects on c_t.

$$\frac{dc_t}{dr} = \frac{[\beta(1+r)]^{1/\sigma}}{(1+r)^2} - \frac{\frac{1}{\sigma}[\beta(1+r)]^{\frac{1}{\sigma}-1}\beta}{1+r} = \frac{[\beta(1+r)]^{1/\sigma}}{(1+r)^2}\left(\frac{\sigma-1}{\sigma}\right)$$

Thus, consumption today increases, $dc_t/dr > 0$ if $\sigma > 1$, i.e. if the IES is low so that the income effect dominates the substitution effect. Consumption today decreases if $\sigma < 1$, i.e. if the IES is high and the substitution effect dominates. If $\sigma = 1$, i.e. for log utility, the income and substitution effects exactly cancel out.